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LETTER TO THE EDITOR

A hierarchical model for scaling structure in generalised diffusion-limited aggregations

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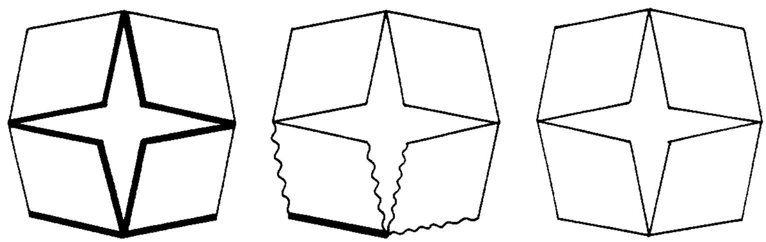
Abstract. Deterministic fractal models are presented to be exactly solvable for the growth probability distribution on the surface of the cluster in a hierarchical lattice. The recursion relations of the electric field on the growth bond are obtained for diffusion-limited aggregation and the dielectric breakdown models. A hierarchy of generalised dimensions $D(q)$ is calculated to describe the growth probability, by using the recursion relations. The partition of $(q-1)D(q)$ into a density of singularities $f(q)$ with singularity strength $\alpha(q)$ is made and the α-f spectra are studied for different dielectric breakdown models. The scaling of the highest growth probability $p_{\max}$ on the growth bond is analytically derived and the dependence of the fractal dimensions is found on the parameter η describing the different dielectric breakdown models.

The essential properties of the kinetic aggregation processes [1-5] are fully described by the growth probability distribution for the perimeter sites (or bonds) of these aggregating clusters [6, 7]. The growth probability can be regarded as a measure associated to each site (bond). The harmonic measure affords a method of quantitatively characterising the relevant properties of the surfaces of such clusters. A hierarchy of generalised dimensions $D(q)$ is used to characterise the harmonic measure, first by Halsey *et al* [6], and independently by Amitrano *et al* [7]. The set of exponents $D(q)$ was first introduced in the context of chaos [8-11] and later to characterise the percolating cluster in a random resistor network [12, 13]. The partition of $D(q)$ into a density of singularities $f(q)$ with singularity strength $\alpha(q)$ is introduced in the context of diffusion-limited aggregation [6]. The α-f spectra are found for the dielectric breakdown models [7]. The growth probability has been measured by a computer experiment [6] and a numerical calculation [7].

In this letter, we present an exactly solvable model for the growth probability on the surface of deterministic fractal aggregates in a hierarchical lattice. We derive the recursion relations of the electric field on the growth bond for the dielectric breakdown models [14] (also referred to as generalised diffusion-limited aggregations). We find the set of generalised dimensions $D(q)$ and the α-f spectrum.

Let us construct the deterministic fractal on a hierarchical lattice. In general, aggregates grown on lattices are viewed as a system of superconductor-normal resistor networks [15] for the dielectric breakdown models. The growth occurs on the perimeter of the aggregate. In these models the growth probability p_i at the growing perimeter bond i is given by $p_i \sim (E_i)^{\eta}$ where E_i is the local electric field at the growth bond.

We merely solve an electrostatic problem for a superconducting cluster inside an infinite normal resistor network. We distinguish between three types of bonds on the lattice: (*a*) superconducting bonds, (*b*) growth bonds which are normal resistors at the perimeter of the aggregate and (*c*) normal resistor bonds except for the growth bonds. Our deterministic fractal model is constructed by the three types of generators shown in figure 1(*a*), (*b*) and (*c*) indicating, respectively, the generators for the superconducting bonds, the growth bonds and the normal resistor bonds. Figure 2 shows the zeroth and first generations. The method of constructing the deterministic fractals on a hierarchical lattice proceeds iteratively. The first generation is obtained from zeroth generation by replacing the growth bond with its generator (shown in figure 1(*b*)). The length scale is transformed by the factor $L_0 = 4$. The second generation is obtained from the first generation in the following way. The superconducting bonds, the growth bonds and the normal resistor bonds (except the growth bonds) are respectively replaced with the three types of generators shown by figure 1(*a*), (*b*) and (*c*). The resultant system is scaled up to four times. The process is continued *ad infinitum.* In this way one can obtain the deterministic fractal aggregate on the hierarchical lattice. We note that the number S ($4 \leqslant S \leqslant 16$) of superconductors within the generator for superconducting bonds (shown in figure 1(*a*)) is an adjustable



(*a*) (*b*) (*c*)

Figure 1. Generators of the deterministic fractal on the hierarchical lattice for kinetic aggregation processes. The superconducting, growth and normal resistor bonds are respectively indicated by the bold, wavy and light lines. (*a*) The generator for the superconducting bonds. (*b*) The generator for the growth bonds. (*c*) The generator for the normal resistor bonds except for the growth bonds.

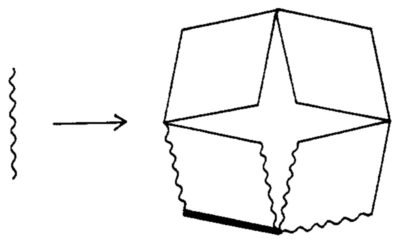


Figure 2. First stage of construction of the deterministic fractal with the use of a line (a growth bond) and the generators in figure 1. The zeroth and first generations are shown on the left-hand and right-hand sides. The first generation is obtained from the zeroth generation by replacing the growth bond with its generator (shown in figure 1(*b*)). The second generation is obtained from the first generation in the way that the superconducting, growth and normal resistor bonds are respectively replaced with the generators shown in figure 1(*a*), (*b*) and (*c*).

parameter which is self-consistently determined by $D(\infty)$. The generator (shown in figure 1(a)) is connected by superconducting bonds from the bottom to the top. The length $L(n)$ and the member $N(n)$ of the superconducting aggregate are given at the nth generation as

$$L(n)=4^{n-1}+4^{n-2}+\ldots+4+1$$

and

$$N(n)=SN(n-1)+4^n$$

and scale as

$$L(n)\sim L_0^n \qquad \text{and} \qquad N(n)\sim S^n. \tag{1}$$

The fractal dimension d_f of the aggregates is given by

$$d_f=\ln S/\ln L_0. \tag{2}$$

Consider the resistance $R(n)$ between the top and bottom at the nth generation. One defines the resistances $R_a(n)$ and $R_b(n)$ on the left-hand and right-hand parts. The following recursion relations are obtained:

$$\begin{aligned} &R(n+1)=1/(1/R_a(n+1)+1/R_b(n+1))\\ &R_a(n+1)=1+1/[1/R(n)+1/(1+R(n))]\\ &R_b(n+1)=1+(1+R(n))/2 \end{aligned} \tag{3}$$

where the initial value $R(0)=1$.

Let us derive the recursion relations for the electric fields on the growth bonds by using the above relations (3). We apply a unit voltage between the top and the bottom. The electric fields $E(1;\, i)$ on the growth bond i at the first generation (see figure 2) are given by

$$\begin{aligned} &E(1;1)=[(R_a(1)-1)/R_a(1)]\\ &E(1;2)=[(R_a(1)-1)/R_a(1)][R(0)/(1+R(0))]\\ &E(1;3)=[(R_b(1)-1)/R_b(1)][R(0)/(1+R(0))]\\ &E(1;4)=E(1;3). \end{aligned} \tag{4}$$

We define the electric fields on the growth bonds at the nth generation by $E(n;\, i_n, i_{n-1},\ldots, i_2, i_1)$. The set $(i_n, i_{n-1},\ldots, i_2, i_1)$ indicates the position of the growth bond where the i_k ranges from 1 to 4. the following relations are obtained:

$$\begin{aligned} &E(n+1;1,i_n,\ldots,i_1)=e_1(n+1)E(n;i_n,\ldots,i_1)\\ &E(n+1;2,i_n,\ldots,i_1)=e_2(n+1)E(n;i_n,\ldots,i_1)\\ &E(n+1;3,i_n,\ldots,i_1)=e_3(n+1)E(n;i_n,\ldots,i_1)\\ &E(n+1;4,i_n,\ldots,i_1)=e_4(n+1)E(n;i_n,\ldots,i_1) \end{aligned} \tag{5}$$

where

$$\begin{aligned} &e_1(n+1)=(R_a(n+1)-1)/R_a(n+1)\\ &e_2(n+1)=[(R_a(n+1)-1)/R_a(n+1)][R(n)/(R(n)+1)]\\ &e_3(n+1)=[(R_b(n+1)-1)/R_b(n+1)][R(n)/(R(n)+1)]\\ &e_4(n+1)=e_3(n+1). \end{aligned}$$

For the maximum electric field $E_{max}(n+1) = E(n+1;\ 1,\ 1,\ldots,1)$, we obtain

$$E_{max}(n+1) = [(R_a(n+1)-1)/R_a(n+1)]E_{max}(n). \tag{6}$$

The growth probability $p(n;\ \eta;\ i_n,\ i_{n-1},\ldots,i_2,\ i_1)$ is given by

$$\begin{aligned} p(n;\ \eta;\ i_n,\ i_{n-1},\ldots,i_2,\ i_1) &\equiv (E(n;\ i_n,\ i_{n-1},\ldots,i_2,\ i_1))^\eta \\ &\quad\times\left(\sum_{i_n=1}^{4}\sum_{i_{n-1}=1}^{4}\cdots\sum_{i_1=1}^{4}(E(n;\ i_n,\ i_{n-1},\ldots,i_2,\ i_1))^\eta\right)^{-1} \\ &= \lambda_{i_n}(n;\ \eta)p(n-1;\ \eta;\ i_{n-1},\ldots,i_2,\ i_1) \end{aligned} \tag{7}$$

where

$$\lambda_{i_n}(n;\ \eta) = e_{i_n}(n)^\eta\left(\sum_{j=1}^{4}e_j(n)^\eta\right)^{-1}.$$

Equation (7) gives a recursion relation between the nth and $(n-1)$th generations. The λ_i represent the growth probability on the bond i within the generator for the growth bond (figure 1(b)). For the maximum growth probability $p_{max}(n;\ \eta)$, we obtain the following recursion relation:

$$p_{max}(n+1;\ \eta) = \lambda_{max}(n+1;\ \eta)p_{max}(n;\ \eta) \tag{8}$$

where

$$\begin{aligned} \lambda_{max}(n+1;\ \eta)^{-1} &= 1+[R(n)/(1+R(n))]^\eta + 2[R_a(n+1)/(R_a(n+1)-1)]^\eta \\ &\quad\times[(R_b(n+1)-1)/R_b(n+1)]^\eta[R(n)/(1+R(n))]^\eta. \end{aligned}$$

For n sufficiently large, the resistances $R_a(n)$, $R_b(n)$ and $R(n)$ approach the fixed point:

$$\begin{aligned} R_a^* &= \lim_{n\to\infty} R_a(n) = 1.5968\ldots \\ R_b^* &= \lim_{n\to\infty} R_b(n) = 1.9377\ldots \\ R^* &= \lim_{n\to\infty} R(n) = 0.8754\ldots. \end{aligned} \tag{9}$$

The growth probabilities λ_i then approach the fixed values λ_i^*. The maximum growth probability $p_{max}(n;\ \eta)$ scales as

$$p_{max}(n;\ \eta) \sim (\lambda_{max}^*(\eta))^n \tag{10}$$

where

$$\lambda_{max}^*(\eta) = \lim_{n\to\infty}\lambda_{max}(n;\ \eta).$$

If one assumes the scaling $p_{max}(\eta) \sim L^{1-d_f}$ [16], the fractal dimension $d_f(\eta)$ is given by

$$d_f(\eta) = 1 - \ln\lambda_{max}^*(\eta)/\ln L_0. \tag{11}$$

The dependence of the fractal dimensions is found on the parameter η describing the different dielectric breakdown models. It is shown in figure 3.

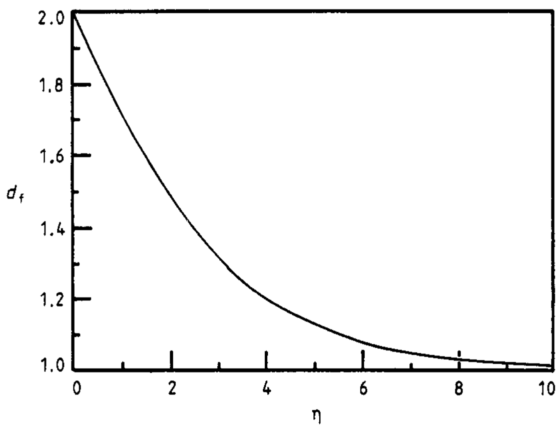


Figure 3. The dependence of the fractal dimensions on the parameter η describing the different dielectric breakdown models.

We can construct an infinite hierarchy of generalised dimensions $D(q;\,\eta)$ which describe the growth probability

$$
\begin{aligned}
D(q;\,\eta) &\equiv \lim_{n\to\infty}(q-1)^{-1}\log\left(\sum_{i_n=1}^{4}\sum_{i_{n-1}=1}^{4}\cdots\sum_{i_1=1}^{4}p(n;\,\eta;\,i_n,i_{n-1},\ldots,i_1)^q\right)(\log L_0^n)^{-1} \\
&= (q-1)^{-1}\log\left(\sum_{i=1}^{4}\lambda_i^*(\eta)^q\right)(\log L_0)^{-1} \qquad (12)
\end{aligned}
$$

where

$$\lambda_i^*(\eta)=\lim_{n\to\infty}\lambda_i(n;\,\eta).$$

We are able to calculate the set of generalised dimensions $D(q;\,\eta)$. For specific values of η, we find $D(\infty;\,0)=1$, $D(\infty;\,1)=0.7099\ldots$ and $D(\infty;\,\infty)=0$ in good agreement with the theoretical prediction $D(\infty;\,\eta)=d_f(\eta)-1$ [6]. The exponents $D(q;\,\eta)$ are plotted in figure 4. Since the number of growth bonds scales as $(L_0)^n$, the $D(0;\,\eta)$

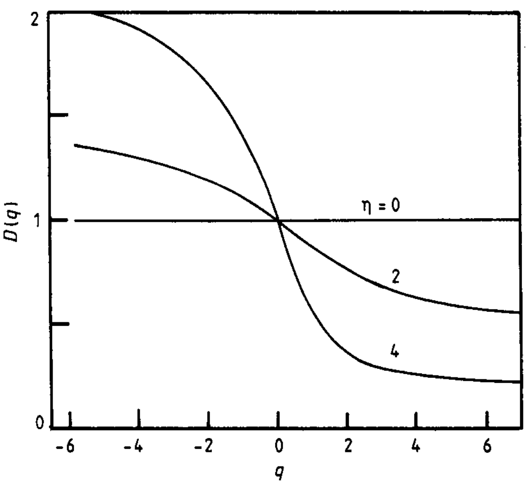


Figure 4. The generalised dimensions $D(q)$ plotted against q for $\eta=0, 2, 4$.

equals one. The information dimension $D(1; 1)$ is less than 1 in contrast to that of the harmonic measure on the square lattice [6, 7]. We find that the shape of $D(q; \eta)$, except $\eta = 0$, is similar to those found for the square lattice [7] and other systems [17]. The partition of $D(q; \eta)$ into a density of singularities $f(q)$ and singularity strength $\alpha(q)$ is introduced:

$$D(q; \eta) = (q-1)^{-1}(q\alpha(q) - f(q)). \quad (13)$$

We display in figure 5 the relation between α and f for the parameter η. The α-f spectra have convex shapes except for $\eta = 0$. In the limiting case $\eta \to 0$ all growth probabilities on the surface are equal, implying the only singularity $\alpha(q) = f(q) = 1$. As the parameter η increases, the α-f spectrum becomes a smoother convex shape. In the opposite limit $\eta \to \infty$ the cluster grows along a straight line $D(\infty; \infty) = 0$.

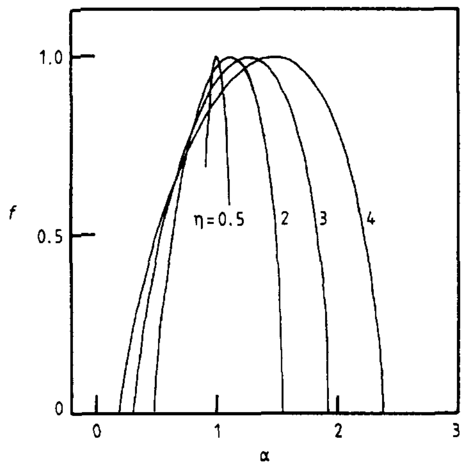


Figure 5. A plot of f against α for $\eta = 0.5, 2, 3, 4$.

In summary, we have presented the deterministic fractal aggregates on the hierarchical lattice in order to analytically calculate growth probability distributions in kinetic aggregation processes. We have analytically found the fractal dimensions and the α-f spectra in diffusion-limited aggregations and related models continuously depending on a parameter η.

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